Lindblad Rate Equations

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In this paper we derive an extra class of non-Markovian master equations where the system state is written as a sum of auxiliary matrixes whose evolution involve Lindblad contributions with local coupling between all of them, resembling the structure of a classical rate equation. The system dynamics may develops strong non-local effects such as the dependence of the stationary properties with the system initialization. These equations are derived from alternative microscopic interactions, such as complex environments described in a generalized Born-Markov approximation and tripartite system-environment interactions, where extra unobserved degrees of freedom mediates the entanglement between the system and a Markovian reservoir. Conditions that guarantees the completely positive condition of the solution map are found. Quantum stochastic processes that recover the system dynamics in average are formulated. We exemplify our results by analyzing the dynamical action of non-trivial structured dephasing and depolarizing reservoirs over a single qubit.

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I. INTRODUCTION

The description of open quantum systems in terms of local in time evolutions is based in a weak coupling and Markovian approximations [1, 2]. When these approximations are valid, the dynamics can be written as a Lindblad equation [1, 2, 3, 4]. The evolution of the density matrix $\rho_S(t)$ of the system of interest reads

$$\frac{d\rho_S(t)}{dt} = \frac{-i}{\hbar}[H_{eff}, \rho_S(t)] - \{D, \rho_S(t)\}_+ + F[\rho_S(t)], \quad (1)$$

where H_{eff} is an effective Hamiltonian, $\{\cdots\}_+$ denotes an anticonmutation operation, and

$$D = \frac{1}{2}\sum_{\alpha,\gamma} a_{\alpha\gamma} V_\gamma^\dagger V_\alpha, \qquad F[\bullet] = \sum_{\alpha,\gamma} a_{\alpha\gamma} V_\alpha \bullet V_\gamma^\dagger. \quad (2)$$

Here, the sum indexes run from one to $(\dim \mathcal{H}_S)^2$, where $\dim \mathcal{H}_S$ is the system Hilbert space dimension. The set $\{V_\alpha\}$ corresponds to a system operator base, and $a_{\alpha\gamma}$ denotes a semipositive Hermitian matrix that characterize the dissipative time scales of the system.

Outside the weak coupling and Markovian approximations, it is not possible to establish a general formalism for dealing with non-Markovian system-environment interactions [5, 6, 7, 8, 9, 10, 11, 12]. Nevertheless, there exist an increasing interest in describing open quantum system dynamics in terms of non-Markovian Lindblad equations [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. Here, the density matrix $\rho_S(t)$ of the system evolves as

$$\frac{d\rho_S(t)}{dt} = \frac{-i}{\hbar}[H_{eff}, \rho_S(t)] + \int_0^t d\tau K(t-\tau)\mathcal{L}[\rho_S(\tau)], \quad (3)$$

where $\mathcal{L}[\bullet] = -\{D, \bullet\}_+ + F[\bullet]$ is a standard Lindblad superoperator. The memory kernel $K(t)$ is a function that may introduces strong non-Markovian effects in the system decay dynamics.

The study and characterization of this kind of dynamics is twofold: on one hand, there is a general fundamental interest in the theory of open quantum systems to extend the methods and concepts well developed for Markovian dynamics to the non-Markov case. On the other hand there are many new physical situations in which the Markov assumption, usually used, is not fulfill and then non-Markovian dynamics has to be introduced. Remarkable examples are single fluorescent systems hosted in complex environments [24, 25, 26, 27, 28], superconducting qubits [29, 30] and band gap materials [31, 32].

Most of the recent analysis on non-Markovian Lindblad evolutions [13, 14, 15, 16, 17, 18, 19] were focus on the possibility of obtaining non-physical solution for $\rho_S(t)$ from Eq. (3). This problem was clarified in Refs. [14, 15], where mathematical constraints on the kernel $K(t)$ that guarantees the completely positive condition [2, 3, 4] of the solution map $\rho_S(0) \to \rho_S(t)$ were found. Furthermore, in Ref. [15] the completely positive condition was associated with the possibility of finding a stochastic representation of the system dynamics.

There also exist different analysis that associate evolutions like Eq. (3) with microscopic system environment interactions [19, 20, 21, 22]. In Ref. [21] the microscopic Hamiltonian involves extra stationary unobserved degrees of freedom that modulate the dissipative coupling between the system of interest an a Markovian environment. This kind of interaction lead to a Lindblad equation characterized by a *random rate*. A similar situation was found in Ref. [22] by considering a complex environment whose action can be described in a *generalized*

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Born-Markov approximation (GBMA). This approach relies in the possibility of splitting the environment in a "direct sum" of sub-reservoirs, each one being able to induce by itself a Markovian system evolution. When the system-environment interaction does not couples the different subspaces associated to each sub-reservoir, the system dynamics can also be written as a Lindblad equation with a random dissipative rate. After performing the average over the random rate, the system dynamics can be written as a non-local evolution with a structure similar to Eq. (3). Besides its theoretical interest, the GBMA was found to be an useful tool for describing and modeling specific physical situations, such as the fluorescence signal scattered by individual nanoscopic systems host in condensed phase environments [28].

The aim of the present work is to go beyond previous results [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], and present an alternative kind of evolution that induces strong non-local effects, providing in this way an extra framework for studying and characterizing non-Markovian open quantum system dynamics. In the present approach, the system density matrix can be written as

$$\rho_S(t) = \sum_R \tilde{\rho}_R(t), \tag{4}$$

where the unnormalized states $\tilde{\rho}_R(t)$ have associated an effective Hamiltonian H_R^{eff}, and their full evolution is defined by

$$\frac{d}{dt}\tilde{\rho}_R(t) = \frac{-i}{\hbar}[H_R^{eff}, \tilde{\rho}_R(t)] - \{D_R, \tilde{\rho}_R(t)\}_+ + F_R[\tilde{\rho}_R(t)] - \sum_{\substack{R' \\ R'\neq R}} \{D_{R'R}, \tilde{\rho}_R(t)\}_+ + \sum_{\substack{R' \\ R'\neq R}} F_{RR'}[\tilde{\rho}_{R'}(t)], \tag{5}$$

subject to the initial conditions

$$\tilde{\rho}_R(0) = P_R \rho_S(0). \tag{6}$$

The positive weights P_R satisfy $\sum_R P_R = 1$. On the other hand, the diagonal superoperator contributions are defined by

$$D_R = \frac{1}{2}\sum_{\alpha,\gamma} a_R^{\alpha\gamma} V_\gamma^\dagger V_\alpha, \qquad F_R[\bullet] = \sum_{\alpha,\gamma} a_R^{\alpha\gamma} V_\alpha \bullet V_\gamma^\dagger, \tag{7}$$

while the non-diagonal contributions reads

$$D_{R'R} = \frac{1}{2}\sum_{\alpha,\gamma} a_{R'R}^{\alpha\gamma} V_\gamma^\dagger V_\alpha, \qquad F_{RR'}[\bullet] = \sum_{\alpha,\gamma} a_{RR'}^{\alpha\gamma} V_\alpha \bullet V_\gamma^\dagger. \tag{8}$$

By convenience, we have introduced different notations for the diagonal and non-diagonal terms. As in standard Lindblad equations, Eq. (1), the matrixes $a_R^{\alpha\gamma}$ and $a_{R'R}^{\alpha\gamma}$ characterize the dissipative rate constants. The structure of the non-diagonal terms in Eq. (5) resemble a classical rate equation [33]. Therefore, we name this kind of evolution as a *Lindblad rate equation.*

Our main objective is to characterize this kind of equations by finding different microscopic interactions that leads to this structure. Furthermore, we find the conditions that guarantees that the solution map $\rho_S(0) \to \rho_S(t)$ is a completely positive one.

While the evolution of $\rho_S(t)$ can be written as a non-local evolution [see Eq. (61)], the structure Eq. (5) leads to a kind of non-Markovian effects where the stationary properties may depend on the system initialization. In order to understand this unusual characteristic, as in Ref. [15, 22], we also explore the possibility of finding a stochastic representation of the system dynamics.

We remark that specific evolutions like Eq. (5) were derived previously in the literature in the context of different approaches [10, 12, 22]. The relation between those results is also clarified in the present contribution.

The paper is organized as follows. Is Sec. II we derive the Lindblad rate equations from a GBMA by considering interactions Hamiltonians that has contribution terms between the subspaces associated to each sub-reservoir. An alternative derivation in terms of tripartite interactions allows to find the conditions under which the dynamic is completely positive. A third derivation is given in terms of quantum stochastic processes. In Sec. III we characterize the resulting non-Markovian master equation. By analyzing some simple non-trivial examples that admits a stochastic reformulation, we explain some non-standard general properties of the non-Markovian dynamics. In Sec. IV we give the conclusions.

II. MICROSCOPIC DERIVATION

In this section we present three alternative situations where the system dynamics is described by a Lindblad rate equation.

A. Generalized Born-Markov approximation

The GBMA applies to complex environments whose action can be well described in terms of a direct sum of Markovian sub-reservoirs [22]. This hypothesis implies that the total system-environment density matrix, in contrast with the standard separable form [1, 2], assumes a classical correlated structure [4] (see Eq. (6) in Ref. [22]). In our previous analysis, we have assumed a system-environment interaction Hamiltonian that does not have matrix elements between the subspaces associated to each sub-reservoir. Therefore it assumes a direct sum structure (see Eq. (5) in Ref. [22]). By raising up this condition, i.e., by taking in account arbitrary interaction Hamiltonians without a direct sum structure, it is possible to demonstrate that the GBMA leads to a Lindblad rate equation, Eq. (5).

As in the standard Born-Markov approximation, the derivation of the system evolution can be formalized in terms of projector techniques [11]. In fact, in Ref. [12]

Breuer and collaborators introduced a "correlated projector technique" intended to describe situations where the total system-environment density matrix does not assume an uncorrelated structure. Therefore, the system dynamics can be alternatively derived in the context of this equivalent approach. The main advantage of this technique is that it provides a rigorous procedure for obtaining the dynamics to any desired order in the system-environment interaction strength [11, 12]. Here, we assume that the system is weakly coupled to the environment. Therefore, we work out the system evolution up to second order in the interaction strength.

We start by considering a full microscopic Hamiltonian description of the interaction of a system S with its environment B

$$H_T = H_S + H_B + H_I. \tag{9}$$

The contributions H_S and H_B correspond to the system and bath Hamiltonians respectively. The term H_I describes their mutual interaction.

The system density matrix follows after tracing out the environment degrees of freedom, $\rho_S(t) = \mathrm{Tr}_B\{\rho_T(t)\}$, where the total density matrix $\rho_T(t)$ evolves as

$$\frac{d\rho_T(t)}{dt} = \frac{-i}{\hbar}[H_T, \rho_T(t)] \equiv \mathcal{L}_T[\rho_T(t)]. \tag{10}$$

Now, we introduce the projector $\mathcal{P}$ defined by

$$\mathcal{P}\rho_T(t) = \sum_R \tilde{\rho}_R(t) \otimes \frac{\Xi_R}{\mathrm{Tr}_B\{\Xi_R\}}, \tag{11}$$

where Ξ_R is given by

$$\Xi_R \equiv \Pi_R \rho_B \Pi_R, \tag{12}$$

with ρ_B being the stationary state of the bath, while the system states $\tilde{\rho}_R(t)$ are defined by

$$\tilde{\rho}_R(t) \equiv \mathrm{Tr}_B\{\Pi_R \rho_T(t) \Pi_R\}. \tag{13}$$

We have introduced a set of projectors $\Pi_R = \sum_{\{\epsilon_R\}} |\epsilon_R\rangle\langle\epsilon_R|$, which provides an orthogonal decomposition of the unit operator $[I_B]$ in the Hilbert space of the bath, $\sum_R \Pi_R = I_B$, with $\Pi_R \Pi_{R'} = \Pi_R \delta_{R,R'}$. The full set of states $|\epsilon_R\rangle$ corresponds to the base where ρ_B is diagonal, which implies $\sum_R \Xi_R = \rho_B$.

It is easy to realize that $\mathcal{P}^2 = \mathcal{P}$. In physical terms, this projector takes in account that each bath-subspace associated to the projectors Π_R induces a different system dynamics, each one represented by the states $\tilde{\rho}_R(t)$. Each sub-space can be seen as a sub-reservoir. On the other hand, notice that the standard projector $\mathcal{P}\rho_T(t) = \mathrm{Tr}_B\{\rho_T(t)\} \otimes \rho_B = \rho_S(t) \otimes \rho_B$ [11], is recuperated when all the states $\tilde{\rho}_R(t)$ have the same dynamics. Therefore, it is evident that the definition of the projector Eq. (11) implies the introduction of a *generalized Born approximation* [22], where instead of a uncorrelated form for the total system-environment density matrix, it is assumed a classical correlated state.

By using that $\sum_R \Pi_R = I_B$, the system density matrix can be written as

$$\rho_S(t) = \sum_R \mathrm{Tr}_B\{\Pi_R \rho_T(t) \Pi_R\} \frac{\mathrm{Tr}_B\{\Xi_R\}}{\mathrm{Tr}_B\{\Xi_R\}} \tag{14a}$$

$$= \mathrm{Tr}_B\{\mathcal{P}\rho_T(t)\} = \sum_R \tilde{\rho}_R(t) \tag{14b}$$

This equation defines the system state as a sum over the states $\tilde{\rho}_R(t)$. Notice that the second line follows from the definition of the objects that define the projector Eq. (11).

By writing the evolution Eq. (10) in an interaction representation with respect to $H_S + H_B$, and splitting the full dynamics in the contributions $\mathcal{P}\rho_T(t)$ and $\mathcal{Q}\rho_T(t)$, where $\mathcal{Q} = 1 - \mathcal{P}$, up to second order in the interaction Hamiltonian it follows [11]

$$\frac{d\mathcal{P}\rho_T(t)}{dt} = \int_0^t dt' \mathcal{P}\mathcal{L}_T(t)\mathcal{L}_T(t')\mathcal{P}\rho_T(t'), \tag{15}$$

where $\mathcal{L}_T(t)$ is the total Liouville superoperator in a interaction representation. For writing the previous equation, we have assumed $\mathcal{Q}\rho_T(0) = 0$, which implies the absence of any initial correlation between the system and the bath, $\rho_T(0) = \rho_S(0) \otimes \rho_B$. Then, the initial condition of each state $\tilde{\rho}_R(t)$ can be written as

$$\tilde{\rho}_R(0) = P_R \rho_S(0). \tag{16}$$

The parameters P_R are defined by the weight of each sub-reservoir in the full stationary bath state

$$P_R = \mathrm{Tr}_B\{\Xi_R\} = \mathrm{Tr}_B\{\Pi_R \rho_B\} = \sum_{\{\epsilon_R\}} \langle\epsilon_R|\rho_B|\epsilon_R\rangle, \tag{17}$$

which trivially satisfies $\sum_R P_R = 1$.

Now, we split the interaction Hamiltonian as

$$H_I = \sum_{R,R'} H_{I_{RR'}} \equiv \sum_{R,R'} \Pi_R H_I \Pi_{R'}. \tag{18}$$

We notice that when $\Pi_R H_I \Pi_{R'} = 0$ for $R \neq R'$, the interaction Hamiltonian can be written as a direct sum $H_I = H_{I_1} \oplus H_{I_2} \cdots \oplus H_{I_R} \oplus H_{I_{R+1}} \cdots$, with $H_{I_R} = \Pi_R H_I \Pi_R$. This case recover the assumptions made in Ref. [22]. In fact, without considering the non-diagonal terms in Eq. (5) $[a^{\alpha\gamma}_{RR'} = 0]$, after a trivial change of notation $\tilde{\rho}_R(t) \to P_R \rho_R(t)$ in Eq. (4), the dynamics reduce to a random Lindblad equation.

In order to proceed with the present derivation, we introduce the superoperator identity [34]

$$[\hat{a}, [\hat{b}, \bullet]] = \frac{1}{2}[[\hat{a}, \hat{b}], \bullet] + \frac{1}{2}\{\{\hat{a}, \hat{b}\}_+, \bullet\}_+ - (\hat{a}\bullet\hat{b} + \hat{b}\bullet\hat{a}), \tag{19}$$

valid for arbitrary operators $\hat{a}$ and $\hat{b}$. By using this identity and the splitting Eq. (18) into Eq. (15), after a straightforward calculation the evolution of $\tilde{\rho}_R(t)$ in the

Schrödinger representation can be written as in Eq. (5). The effective Hamiltonians read

$$H_R^{eff} = H_S - i\frac{\hbar}{2}\int_0^\infty d\tau \mathrm{Tr}_{B_R}\{[H_I, H_I(-\tau)]\rho_{B_R}\}. \quad (20)$$

The non-diagonal operators $D_{R'R}$ read

$$D_{R'R} = \frac{1}{2}\int_0^\infty d\tau \mathrm{Tr}_{B_R}([H_{I_{RR'}} H_{I_{R'R}}(-\tau) + h.c.]\rho_{B_R}), \quad (21)$$

while the corresponding superoperators $F_{RR'}$ can be written as

$$F_{RR'}[\bullet] = \int_0^\infty d\tau \mathrm{Tr}_{B_R}(H_{I_{RR'}}(-\tau)[\bullet]\otimes\rho_{B_{R'}} H_{I_{R'R}} + h.c.). \quad (22)$$

The diagonal contributions follows from the previous expressions as $D_R = D_{RR}$, and $F_R[\bullet] = F_{RR}[\bullet]$. Furthermore, we have defined $\mathrm{Tr}_{B_R}\{\bullet\} \equiv \mathrm{Tr}_B\{\Pi_R \bullet \Pi_R\}$ and

$$\rho_{B_R} \equiv \Xi_R / P_R. \quad (23)$$

Notice that these objects correspond to the stationary state of each sub-reservoir.

In obtaining Eqs. (20) to (22) we have introduced a standard *Markovian approximation* [1, 2], which allows to obtain local in time evolutions for the set $\{\tilde{\rho}_R(t)\}$, as well as to extend the time integrals to infinite. This approximation applies when the diagonal and non-diagonal correlations of the different sub-reservoirs define the small time scale of the problem. In order to clarify the introduction of the Markov approximation, we assume that the interaction Hamiltonian can be written as

$$H_I = \sum_\alpha V_\alpha \otimes B_\alpha, \quad (24)$$

where the operators V_α and B_α act on the system and bath Hilbert spaces respectively. By using $H_I = H_I^\dagger$, the previous expressions Eqs. (21) and (22) read

$$D_{R'R} = \frac{1}{2}\sum_{\alpha\beta}\int_0^\infty d\tau\{\chi^{\alpha\beta}_{R'R}(-\tau) V_\alpha^\dagger V_\beta(-\tau) + h.c.\}, \quad (25)$$

and

$$F_{RR'}[\bullet] = \sum_{\alpha\beta}\int_0^\infty \{d\tau \chi^{\alpha\beta}_{RR'}(-\tau) V_\beta(-\tau)[\bullet]V_\alpha^\dagger + h.c.\}. \quad (26)$$

Here, we have defined the "*projected bath correlations*"

$$\chi^{\alpha\beta}_{RR'}(-\tau) \equiv \mathrm{Tr}_{B_{R'}}\{\rho_{B_{R'}} B_\alpha^\dagger \Pi_R B_\beta(-\tau)\}. \quad (27)$$

Without taking in account the indexes R and R', this expression reduces to the standard definition of bath correlation [1, 2, 3, 34]. Here, the same structure arises with projected elements. As the integrals that appears in Eqs. (25) and (26) have the same structure that in the standard Born-Markov approximation [34], the meaning of the previous calculation steps becomes clear.

Finally, in order to obtain the explicit expressions for the matrixes $a^{\alpha\gamma}_{RR'}$ and $a^{\alpha\gamma}_R$, we define a matrix $C_{\beta\gamma}(-\tau)$ from

$$V_\beta(-\tau) = e^{-i\tau H_S} V_\beta e^{+i\tau H_S} = \sum_\gamma C_{\beta\gamma}(-\tau) V_\gamma. \quad (28)$$

By introducing these coefficients in Eqs. (25) and (26), it is possible to write the operators $D_{R'R}$ and $F_{RR'}[\bullet]$ as in Eq. (8). The matrix $a^{\alpha\gamma}_{RR'}$ is defined by

$$a^{\alpha\gamma}_{RR'} = \sum_\beta \int_0^\infty d\tau \chi^{\gamma\beta}_{RR'}(-\tau)\ C_{\beta\alpha}(-\tau) + \sum_\beta \int_0^\infty d\tau (\chi^{\alpha\beta}_{RR'})^*(-\tau)\ C^*_{\beta\gamma}(-\tau), \quad (29)$$

while the diagonal matrix elements follows as $a^{\alpha\gamma}_R = a^{\alpha\gamma}_{RR}$. Consistently, without taking in account the indexes R and R', this matrix structure reduce to that of the standard Born-Markov approximation [34].

Quantum master equation for a system influencing its environment

In Ref. [10], Esposito and Gaspard deduced a quantum master equation intended to describe physical situations where the density of states of a reservoir is affected by the changes of energy of an open system. While this physical motivation is different to that of the GBMA [22] (or in general, to the correlated projector techniques [12]), here we show that both formalisms can be deduced by using the same calculations steps. Therefore, the evolution of Ref. [10] can also be written as a Lindblad rate equation.

In Ref. [10], the system evolution is derived by taking in account the effect of the energy exchanges between the system and the environment and the conservation of energy by the total (closed) system-reservoir dynamics. These conditions are preserved by tracing-out the bath coherences and maintaining all the information with respect to the bath populations. Therefore, the system density matrix is written in terms of an auxiliary state that depends parametrically on the energy of the environment, which is assumed in a microcanonical state. By noting that in the GBMA there not exist any coherence between the different sub-reservoirs [see Eq. (11)], we realize that the dynamics obtained in Ref. [10] can be recovered with the previous results by associating the *discrete* index R with a *continuos* parameter ε, which label the eigenvalues of the reservoir, joint with the replacements

$$\tilde{\rho}_R(t) \to \tilde{\rho}(\varepsilon; t), \qquad \sum_R \to \int d\varepsilon\ n(\varepsilon), \quad (30)$$

where $n(\varepsilon)$ is the spectral density function of the reservoir. Consistently, the system state [Eq. (14)] is written as

$$\rho_S(t) = \int d\varepsilon\ n(\varepsilon)\tilde{\rho}(\varepsilon; t) \equiv \int d\varepsilon\ \bar{\rho}(\varepsilon; t). \quad (31)$$

As in the GBMA, the evolution of $\bar{\rho}(\varepsilon;t)$ can be written as a Lindblad rate equation defined in terms of the matrix structure Eq. (29) with the replacement $\chi^{\alpha\beta}_{RR'}(-\tau) \to \chi^{\alpha\beta}_{\varepsilon\varepsilon'}(-\tau)$, where

$$\chi^{\alpha\beta}_{\varepsilon\varepsilon'}(-\tau) = \langle\varepsilon'|\, B^{\dagger}_{\alpha}\,|\varepsilon\rangle\,\langle\varepsilon|\, B_{\beta}\,|\varepsilon'\rangle \exp[-i(\varepsilon-\varepsilon')\tau]. \quad (32)$$

This last definition follows from the microcanonical state of the reservoir [$\rho_B \to 1$]. Finally, by introducing the matrix elements

$$P_{ss'}(\varepsilon;t) \equiv \langle s|\bar{\rho}(\varepsilon;t)|s'\rangle, \quad (33)$$

where $\{|s\rangle\}$ are the eigenstates of the system Hamiltonian, $H_S|s\rangle = \varepsilon_s|s\rangle$, the master equation of Ref. [10] is explicitly recovered. Due to the energy preservation condition, in general the evolution involves a continuos parametric coupling between the matrix elements $P_{ss'}(\varepsilon;t)$ and $P_{ss'}(\varepsilon \pm \Delta;t)$, where Δ is a energy scale that characterize the natural transition frequencies of the system [10].

We remark that the difference between both approaches relies on the assumed properties of the environment. In the context of the GBMA, the index R label a set of Hilbert subspaces each one defined in terms of a manifold of bath eigenstates able to induce, by itself, a Markovian system dynamics. Therefore, by hypothesis, the complete environment does not feels the effects of the system energy changes. On the other hand, the approach of Esposito and Gaspard applies to the opposite situation where, by hypothesis, the density of states of the environment vary on a scale comparable to the system energy transitions. The stretched similarity between both approaches follows from the absence of coherences between the different (discrete or continuous) bath subspaces. In both cases the system evolution can be written as a Lindblad rate equation.

B. Composite environments

The previous analysis relies in a bipartite system-environment interaction described in a GBMA. Here, we arrive to a Lindblad rate equation by considering composite environments, where extra degrees of freedom U modulate the interaction (the entanglement) between a system S and a Markovian reservoir B [21]. This formulation allows to find the conditions under which Eq. (5) defines a completely positive evolution.

The total Hamiltonian reads

$$H_T = H_S + H_U + H_{SU} + H_B + H_I. \quad (34)$$

As before, H_S represent the system Hamiltonian. Here, H_B is the Hamiltonian of the Markovian environment. On the other hand, H_U is the Hamiltonian of the extra degrees of freedom that modulate the system-environment interaction. The interaction Hamiltonian H_I couples the three involved parts. We also consider the possibility of a direct interaction between S and U, denoted by H_{SU}.

As B is a Markovian reservoir, we can trace out its degrees of freedom in a standard way [1, 2, 3]. Therefore, we assume the completely positive Lindblad evolution

$$\frac{d\rho_C(t)}{dt} = \frac{-i}{\hbar}[H_C, \rho_C(t)] - \{D_C, \rho_C(t)\}_+ + F_C[\rho_C(t)], \quad (35)$$

with the definitions

$$D_C = \frac{1}{2}\sum_{i,j} b_{ij} A^{\dagger}_j A_i, \qquad F_C[\bullet] = \sum_{i,j} b_{ij} A_i \bullet A^{\dagger}_j. \quad (36)$$

The matrix $\rho_C(t)$ corresponds to the state of the "compose system" S-U with Hilbert space $\mathcal{H}_C = \mathcal{H}_S \otimes \mathcal{H}_U$. The sum indexes i and j run from one to 1 to $(\dim \mathcal{H}_C)^2$, with $\dim \mathcal{H}_C = \dim \mathcal{H}_S \dim \mathcal{H}_U$. Consistently, the set $\{A_i\}$ is a base of operators in $\mathcal{H}_C$, and b_{ij} is an arbitrary Hermitian semipositive matrix.

In order to get the system state it is also necessary to trace out the degrees of freedom U. In fact, $\rho_S(t) = \mathrm{Tr}_U\{\rho_C(t)\}$, which deliver

$$\begin{aligned} \rho_S(t) &= \mathrm{Tr}_U\{\rho_C(t)\} = \sum_R \langle R|\, \rho_C(t)\, |R\rangle\,, \\ &\equiv \sum_R \tilde{\rho}_R(t). \end{aligned} \quad (37)$$

where $\{|R\rangle\}$ is a base of vector states in $\mathcal{H}_U$. We notice that here, the sum structure Eq. (4) have a trivial interpretation in terms of a trace operation.

By assuming an uncorrelated initial condition $\rho_C(0) = \rho_S(0)\otimes\rho_U(0)$, where $\rho_S(0)$ and $\rho_U(0)$ are arbitrary initial states for the systems S and U, from Eq. (37) it follows the initial conditions $\tilde{\rho}_R(0) = P_R\rho_S(0)$, where

$$P_R = \langle R|\, \rho_U(0)\, |R\rangle\,. \quad (38)$$

Therefore, here the weights P_R corresponding to Eq. (6) are defined by the diagonal matrix elements of the initial state of the system U. From now on, we will assume that the set of states $\{|R\rangle\}$ correspond to the eigenvectors basis of H_U, i.e.,

$$H_U\,|R\rangle = \varepsilon_R\,|R\rangle\,. \quad (39)$$

The evolution of the states $\tilde{\rho}_R(t) = \langle R|\, \rho_C(t)\, |R\rangle$ can be obtained from Eq. (35) after tracing over system U. Under special *symmetry conditions*, the resulting evolution can be cast in the form of a Lindblad rate equation, Eq. (5). In fact, in a general case, there will be extra contributions proportional to the components $\langle R|\, \rho_C(t)\, |R'\rangle$. By noting that

$$\mathrm{Tr}_S[\langle R|\, \rho_C(t)\, |R'\rangle] = \langle R|\, \rho_U(t)\, |R'\rangle\,, \quad (40)$$

where $\rho_U(t) = \mathrm{Tr}_S\{\rho_C(t)\}$ is the density matrix of the degrees of freedom U, we realize that the evolution of $\tilde{\rho}_R(t)$ can be written as a Lindblad rate equation only when the

evolution of $\rho_U(t)$ does not involves coupling between the populations $\langle R|\,\rho_U(t)\,|R\rangle$ and coherences $\langle R|\,\rho_U(t)\,|R'\rangle$, $R \neq R'$, of system U. As is well known [1, 2, 3], this property is satisfied when the dissipative evolution of $\rho_U(t)$ can be written in terms of the eigenoperators L_u of the unitary dynamic, i.e., $[H_U, L_u] = \omega_u L_u$. In what follows, we show explicitly that this property is *sufficient* to obtain a Lindblad rate equation for the set of matrixes $\{\tilde{\rho}_R(t)\}$.

First, we notice that the Hamiltonian H_C in Eq. (35) must to have the structure

$$H_C = H_S + H_U + \sum_\alpha V_\alpha \otimes L_0^\alpha, \tag{41}$$

where L_0^α are the eigenoperators with a null eigenvalue, i.e., $[H_U, L_0^\alpha] = 0$. With this structure, the populations and coherences corresponding to U do not couple between them. Therefore, the effective Hamiltonian H_R^{eff} in Eq. (5) reads

$$H_R^{eff} = H_S + \sum_\alpha \langle R|\,L_0^\alpha\,|R\rangle\, V_\alpha. \tag{42}$$

After taking the operator base in $\mathcal{H}_C = \mathcal{H}_S \otimes \mathcal{H}_U$ as

$$\{A_i\} \to \{V_\alpha \otimes L_u\}, \tag{43}$$

the superoperators Eq. (36) can be written as

$$D_C = \frac{1}{2}\sum_{\substack{\alpha,\gamma\\ u,v}} b_{uv}^{\alpha\gamma} V_\gamma^\dagger L_v^\dagger V_\alpha L_u, \tag{44a}$$

$$F_C[\bullet] = \sum_{\substack{\alpha,\gamma\\ u,v}} b_{uv}^{\alpha\gamma} V_\alpha L_u \bullet V_\gamma^\dagger L_v^\dagger. \tag{44b}$$

With these definitions, by taking the trace operation over the system U in the evolution Eq. (35), we notice that the evolution of the set $\{\tilde{\rho}_R(t)\}$ can be cast in the form of a Lindblad rate equation if the conditions

$$\sum_{u,v} b_{uv}^{\alpha\gamma}\,\langle R''|\,L_v^\dagger\,|R\rangle\,\langle R|\,L_u\,|R'\rangle = \delta_{R',R''}\,a_{RR'}^{\alpha\gamma} \tag{45}$$

are satisfied. The factor $\delta_{R',R''}$ guarantees that the evolution of the set $\{\tilde{\rho}_R(t)\}$ do not involve the terms $\langle R|\,\rho_C(t)\,|R'\rangle$, $R \neq R'$, and in turn implies that the populations and coherences of U do not couple between them. On the other hand, $a_{RR'}^{\alpha\gamma}$ defines the matrix elements corresponding to the structure Eq. (5). The diagonal contributions follows from Eq. (45) by taking $R = R'$.

The set of conditions Eq. (45) can be simplified by taking the base

$$L_u \to |\mathcal{R}'\rangle\langle\mathcal{R}|, \tag{46}$$

which from Eq. (39) satisfy $[H_U, L_u] = (\varepsilon_{\mathcal{R}} - \varepsilon_{\mathcal{R}'})L_u$. Thus, Eq. (45) can be consistently satisfied if we impose

$$b_{uv}^{\alpha\gamma} = 0, \qquad for \qquad u \neq v. \tag{47}$$

After changing $\sum_u \to \sum_{\mathcal{R},\mathcal{R}'}$ in Eq. (45), we get

$$a_{RR'}^{\alpha\gamma} = b_{(R,R')(R,R')}^{\alpha\gamma}, \qquad a_R^{\alpha\gamma} = b_{(R,R)(R,R)}^{\alpha\gamma}, \tag{48}$$

where we have used that $\mathcal{R}$ and $\mathcal{R}'$ are dumb indexes. This result demonstrate that the evolution induced by the composite environment can in fact be written as a Lindblad rate evolution Eq. (5) with the matrix elements defined by Eq. (48).

From our previous considerations we deduce that Lindblad rate equation arise from microscopic tripartite interactions having the structure

$$H_I = L_0 \otimes H_{SB} + \sum_u L_u \otimes H_{SB}^u + L_u^\dagger \otimes (H_{SB}^u)^\dagger, \tag{49}$$

where $[H_U, L_0] = 0$, and $L_u \to |R\rangle\langle R'|$ with $R \neq R'$. On the other hand, H_{SB}^u are arbitrary interaction terms between the system S and the Markovian environment B. In fact, the structure Eq. (49) guarantees that the populations and coherences of U do not couple between them, which in turn implies that the evolutions of the system S is given by a Lindblad rate equation.

Completely positive condition

We have presented two different microscopic interactions that lead to a Lindblad rate equation. In order to use these equations as a valid tool for modeling open quantum system dynamics it is necessary to establish the conditions under which the solution map $\rho_S(0) \to \rho_S(t)$ is a completely positive one. For an arbitrary Lindblad rate equation this condition must to be defined in terms of the matrixes $a_{RR'}^{\alpha\gamma}$ and $a_R^{\alpha\gamma}$.

In order to find the allowed matrix structures, we notice that the evolution Eq. (35) is a completely positive one when $b_{ij} \to b_{(R,R')(R,R')}^{\alpha\gamma}$ is a semipositive defined matrix. Therefore, by using Eq. (48) we arrive to the conditions

$$|a_{RR'}^{\alpha\gamma}| \geq 0, \qquad |a_R^{\alpha\gamma}| \geq 0, \qquad \forall\, R, R', \tag{50}$$

i.e., for any value of R and R' both kind of matrixes must to be semipositive defined in the system indexes α, γ. The condition $|a_R^{\alpha\gamma}| \geq 0$ has a trivial interpretation. In fact, when $a_{RR'}^{\alpha\gamma} = 0$, there not exist any dynamical coupling between the states $\tilde{\rho}_R(t)$. Thus, their evolutions are defined by a Lindblad structure that under the constraint $|a_R^{\alpha\gamma}| \geq 0$ define a completely positive evolution.

C. Quantum random walk

By using the similarity of Eq. (5) with a classical rate equation [33], here we present a third derivation by constructing a stochastic dynamics that develops in the system Hilbert space and whose average evolution is given by a Lindblad rate equation.

First, we assume that the system is endowed with a classical internal degree of freedom characterized by a set $\{R\}$ of possible states. The corresponding populations $P_R(t)$ obey the classical evolution

$$\frac{dP_R(t)}{dt} - \sum_{\substack{R' \\ R' \neq R}} \gamma_{R'R} P_R(t) + \sum_{\substack{R' \\ R' \neq R}} \gamma_{RR'} P_{R'}(t), \qquad (51)$$

with initial conditions $P_R(0) = P_R$, and where the coefficients $\{\gamma_{R'R}\}$ define the hopping rates between the different classical states R.

To each state R we associate a different Markovian system dynamics, whose evolution is generated by the superoperator

$$\bar{\mathcal{L}}_R = \mathcal{L}_H + \mathcal{L}_R, \qquad (52)$$

with $\mathcal{L}_H[\bullet] = (-i/\hbar)[H_S, \bullet]$ and a standard Lindblad contribution $\mathcal{L}_R[\bullet] = -\{D_R, \bullet\}_+ + F_R[\bullet]$. Therefore, each state R defines a *propagation channel* with a different self-dynamic. The system state follows by tracing out any information about the internal state. Thus, we write

$$\rho_S(t) = \sum_R \tilde{\rho}_R(t), \qquad (53)$$

where each state $\tilde{\rho}_R(t)$ defines the system state *given* that the internal degree of freedom is in the state R. Consistently, the initial condition of the auxiliary states reads $\tilde{\rho}_R(0) = P_R \rho_S(0)$.

Finally, we assume that in each transition $R \to R'$ of the internal degree of freedom, it is applied a completely positive superoperator $\mathcal{E}_R$ [2, 3, 4], which produces a disruptive transformation in the system state.

The stochastic dynamics is completely defined after providing the self-channel dynamics, defined by $\{\bar{\mathcal{L}}_R\}$, the set of rates $\{\gamma_{R'R}\}$ and the superoperators $\{\mathcal{E}_R\}$. By construction this dynamics is completely positive. The explicit construction of the corresponding stochastic realizations, which develop in the system Hilbert space, is as follows. When the system is effectively in channel R, it is transferred to channel R' with rate $\gamma_{R'R}$. Therefore, the probability of staying in channel R during a sojourn interval t is given by

$$P_0^{(R)}(t) = \exp[-t \sum_{\substack{R' \\ R' \neq R}} \gamma_{R'R}]. \qquad (54)$$

This function completely defines the statistics of the time intervals between the successive disruptive events. As in standard classical rate equations, when the system "jump outside" of channel R, each subsequent channel R' is selected with probability

$$t_{R'R} = \frac{\gamma_{R'R}}{\sum_{\substack{R'' \\ R'' \neq R}} \gamma_{R''R}}, \qquad (55)$$

in such a way that $\sum_{R'} t_{R'R} = 1$. Furthermore, each transference $R \to R'$, is attended by the application of the superoperator $\mathcal{E}_R$, which produces the disruptive transformation $\tilde{\rho}_R(t) \to \mathcal{E}_R[\tilde{\rho}_R(t)]$. This transformed state is the subsequent initial condition for channel R'.

The average over realizations of the previous quantum stochastic process, for each state $\tilde{\rho}_R(t)$, reads

$$\tilde{\rho}_R(t) = P_0^{(R)}(t) e_R^{t\bar{\mathcal{L}}_R} \tilde{\rho}(0) + \int_0^t d\tau P_0^{(R)}(t-\tau) e^{(t-\tau)\bar{\mathcal{L}}_R} \times \sum_{\substack{R' \\ R' \neq R}} \gamma_{RR'} \mathcal{E}_{R'}[\tilde{\rho}_{R'}(\tau)], \qquad (56)$$

The structure of this equation has a clear interpretation. The first contribution represents the realization where the system remains in channel R without happening any scattering event. Clearly this term must be weighted by the probability of not having any event in the time interval $(t, 0)$, i.e., with the probability $P_0^{(R)}(t)$. On the other hand, the terms inside the integral correspond to the rest of the realizations. They take in account the contributions that come from any other channel R', arriving at time τ and surviving up to time t in channel R. During this interval it is applied the self-channel propagator $\exp[(t-\tau)\bar{\mathcal{L}}_R]$. As before, this evolution is weighted by the survival probability $P_0^{(R)}(t-\tau)$.

By working Eq. (56) in the Laplace domain, after a simple calculation, it is possible to arrive to the evolution

$$\frac{d}{dt}\tilde{\rho}_R(t) = \frac{-i}{\hbar}[H_S, \tilde{\rho}_R(t)] - \{D_R, \tilde{\rho}_R(t)\}_+ + F_R[\tilde{\rho}_R(t)] - \sum_{\substack{R' \\ R' \neq R}} \gamma_{R'R} \tilde{\rho}_R(t) + \sum_{\substack{R' \\ R' \neq R}} \gamma_{RR'} \mathcal{E}_{R'}[\tilde{\rho}_{R'}(t)]. \qquad (57)$$

We notice that this expression does not corresponds to the more general structure of a Lindblad rate equation, Eq. (5). Nevertheless, there exist different non-trivial situations that fall in this category. As we demonstrate in the next section, the advantage of this formulation is that it provides a simple framework for understanding some non-usual characteristics of the system dynamics.

III. NON-MARKOVIAN DYNAMICS

In this section we obtain the master equation that define the evolution of the system state $\rho_S(t)$ associated to an arbitrary Lindblad rate equation, Eq. (5).

In order to simplify the notation, we define a column vector defined in the R-space and whose elements are the states $\tilde{\rho}_R$, i.e., $|\tilde{\rho}) = (\tilde{\rho}_1, \tilde{\rho}_2, \ldots \tilde{\rho}_R, \ldots)^{\mathrm{T}}$, where T denote a transposition operation. Then, the evolution Eq. (5) can be written as

$$\frac{d\,|\tilde{\rho}(t))}{dt} = \mathcal{L}_H\,|\tilde{\rho}(t)) + \hat{\mathbb{M}}\,|\tilde{\rho}(t))\,. \qquad (58)$$

where $\mathcal{L}_H[\bullet] = -(i/\hbar)[H_S, \bullet]$, and the matrix elements of $\hat{\mathbb{M}}$ reads

$$\hat{\mathbb{M}}_{RR'}[\bullet] = \delta_{R,R'}\left\{\frac{-i}{\hbar}[H'_R, \bullet] - \{D_R, \bullet\}_+ + F_R[\bullet]\right\} + F_{RR'}[\bullet] - \delta_{R,R'}\sum_{\substack{R'' \\ R''\neq R}}\{D_{R''R}, \bullet\}_+, \quad (59)$$

where $H'_R = H^{eff}_R - H_S$, is the shift Hamiltonian produced by the interaction with the reservoir. The initial condition reads $|\tilde{\rho}(0)) = |P)\,\rho_S(0)$, where we have introduced the vector $|P) = (P_1, P_2, \ldots P_R, \ldots)^{\mathrm{T}}$. The system state Eq. (4) reads $\rho_S(t) = (1\,|\tilde{\rho}(t))$, where $|1)$ is the row vector with elements equal to one. Notice that due to the normalization of the statistical weights it follows $(1\,|P) = 1$.

From Eq. (58), the system state can be trivially written in the Laplace domain as

$$\rho_S(u) = (1|\,\frac{1}{u - (\mathcal{L}_H + \hat{\mathbb{M}})}\,|P)\,\rho_S(0), \quad (60a)$$

$$\equiv (1|\,\hat{\mathbb{G}}(u)\,|P)\,\rho_S(0), \quad (60b)$$

where u is the conjugate variable. Multiplying the right term by the identity operator written in the form $1/\,(1|\,\hat{\mathbb{G}}(u)[u - (\mathcal{L}_H + \hat{\mathbb{M}})]\,|P)$, it is straightforward to arrive to the non-local evolution

$$\frac{d\rho_S(t)}{dt} = \mathcal{L}_H[\rho_S(t)] + \int_0^t d\tau\,\mathbb{L}(t-\tau)[\rho_S(\tau)], \quad (61)$$

where the superoperator $\mathbb{L}(t)$ is defined by the relation

$$(1|\,\hat{\mathbb{G}}(u)\hat{\mathbb{M}}\,|P)\,[\bullet] = (1|\,\hat{\mathbb{G}}(u)\,|P)\,\mathbb{L}(u)[\bullet]. \quad (62)$$

In general, depending on the underlying structure, the evolution Eq. (61) involves many different memory kernels, each one associated to a Lindblad contribution.

We notice that a similar master equation was obtained in Refs. [21, 22]. Nevertheless, here the dynamics may strongly departs with respect to the evolutions that arise from Lindblad equations with a random rate $[a^{\alpha\gamma}_{RR'} = 0]$. In fact, the previous calculation steps are valid only if

$$\lim_{u\to 0}(1|\,u\hat{\mathbb{G}}(u)\,|P) = 0. \quad (63)$$

By using that $\lim_{t\to\infty} f(t) = \lim_{u\to 0} uf(u)$, this condition is equivalent to $\lim_{t\to\infty}(1|\,\hat{\mathbb{G}}(t)\,|P) = 0$. In the general case $a^{\alpha\gamma}_{RR'} \neq 0$, Eq. (63) is not always satisfied. In this situation, the density matrix evolution becomes non-homogenous and the stationary state may depends on the system initial condition. In general, this case may arises when the diagonal contributions are null, i.e., $a^{\alpha\gamma}_{R} = 0$ and $a^{\alpha\gamma}_{RR'} \neq 0$. We remark that these matrix structures values are completely consistent with the conditions Eq. (50). On the other hand, in the context of the GBMA, this case arise when the diagonal sub-bath correlations are null, $\chi^{\alpha\beta}_{RR}(-\tau) = 0$, which in turn implies that the interaction Hamiltonian Eq. (18) satisfies $\Pi_R H_I \Pi_{R'} = 0$ if $R = R'$.

In order to characterize the dynamics when the condition Eq. (63) is not satisfied, we introduce the difference

$$\delta\rho_S(u) \equiv \rho_S(u) - \frac{1}{u}\lim_{u\to 0}(1|\,u\hat{\mathbb{G}}(u)\,|P)\,\rho_S(0), \quad (64a)$$

$$= (1|\,\hat{\mathbb{G}}(u) - \frac{1}{u}\lim_{u\to 0} u\hat{\mathbb{G}}(u)\,|P)\,\rho_S(0), \quad (64b)$$

$$\equiv (1|\,\delta\hat{\mathbb{G}}(u)\,|P)\,\rho_S(0), \quad (64c)$$

where now the pseudo-propagator $\delta\hat{\mathbb{G}}(u)$ satisfies $\lim_{u\to 0}(1|\,u\delta\hat{\mathbb{G}}(u)\,|P) = 0$. Therefore, $\delta\rho_S(t)$ satisfies an evolution like Eq. (61) where the kernel is defined by Eq. (62) with $\hat{\mathbb{G}}(u) \to \delta\hat{\mathbb{G}}(u)$. Notice that the system state, even in the stationary regime, involves the contribution $\lim_{u\to 0}(1|\,u\hat{\mathbb{G}}(u)\,|P)\,\rho_S(0)$, that in fact depends on the system initial condition.

In the next examples we show the meaning of this property, as well as its interpretation in the context of the stochastic approach.

A. Dephasing environment

Here we analyze the case of a qubit system interacting with a dispersive reservoir [4, 15] whose action can be written in terms of a dispersive Lindblad rate equation. We assume a complex reservoir with only two subspaces, $R = a, b$, whose statistical weights [Eq. (17)] satisfy $P_a + P_b = 1$. Thus, the system state reads

$$\rho_S(t) = \tilde{\rho}_a(t) + \tilde{\rho}_b(t). \quad (65)$$

A generalization to an arbitrary number of sub-reservoir is straightforward.

The evolution of the auxiliary states are taken as

$$\frac{d}{dt}\tilde{\rho}_a(t) = -\gamma_a[\tilde{\rho}_a(t) - \sigma_z\tilde{\rho}_a(t)\sigma_z] - \gamma_{ba}\tilde{\rho}_a(t) + \gamma_{ab}\sigma_z\tilde{\rho}_b(t)\sigma_z, \quad (66a)$$

$$\frac{d}{dt}\tilde{\rho}_b(t) = -\gamma_b[\tilde{\rho}_b(t) - \sigma_z\tilde{\rho}_b(t)\sigma_z] - \gamma_{ab}\tilde{\rho}_b(t) + \gamma_{ba}\sigma_z\tilde{\rho}_a(t)\sigma_z, \quad (66b)$$

where σ_z is the z Pauli matrix. The completely positive conditions Eq. (50) imply

$$\gamma_a \geq 0, \qquad \gamma_b \geq 0, \quad (67a)$$

$$\gamma_{ab} \geq 0, \qquad \gamma_{ba} \geq 0. \quad (67b)$$

By denoting the matrix elements by $(R = a, b)$

$$\tilde{\rho}_R(t) = \begin{pmatrix} \Pi^+_R(t) & \Phi^+_R(t) \\ \Phi^-_R(t) & \Pi^-_R(t) \end{pmatrix}, \quad (68)$$

the evolution corresponding to the populations read

$$\frac{d}{dt}\Pi_a^{\pm}(t) = -\gamma_{ba}\Pi_a^{\pm}(t) + \gamma_{ab}\Pi_b^{\pm}(t), \quad (69a)$$

$$\frac{d}{dt}\Pi_b^{\pm}(t) = -\gamma_{ab}\Pi_b^{\pm}(t) + \gamma_{ba}\Pi_a^{\pm}(t), \quad (69b)$$

with $\Pi_R^{\pm}(0) = P_R\Pi_S^{\pm}(0)$, while for the coherences we obtain

$$\frac{d}{dt}\Phi_a^{\pm}(t) = -(\gamma_a + \gamma_{ba})\Phi_a^{\pm}(t) - \gamma_{ab}\Phi_b^{\pm}(t), \quad (70a)$$

$$\frac{d}{dt}\Phi_b^{\pm}(t) = -(\gamma_b + \gamma_{ab})\Phi_b^{\pm}(t) - \gamma_{ba}\Phi_a^{\pm}(t), \quad (70b)$$

with $\Phi_R^{\pm}(0) = P_R\Phi_S^{\pm}(0)$. For expressing the initial conditions we have trivially extended the notation Eq. (68) to the matrix elements of $\rho_S(t)$.

We notice that all coherences and populations evolve independently each of the others. From the evolution of the populations Eq. (69) it follow

$$\frac{d}{dt}\mathrm{Tr}[\tilde{\rho}_a(t)] = -\gamma_{ba}\mathrm{Tr}[\tilde{\rho}_a(t)] + \gamma_{ab}\mathrm{Tr}[\tilde{\rho}_b(t)], \quad (71a)$$

$$\frac{d}{dt}\mathrm{Tr}[\tilde{\rho}_b(t)] = -\gamma_{ab}\mathrm{Tr}[\tilde{\rho}_b(t)] + \gamma_{ba}\mathrm{Tr}[\tilde{\rho}_a(t)], \quad (71b)$$

with $\mathrm{Tr}[\tilde{\rho}_a(0)] + \mathrm{Tr}[\tilde{\rho}_b(0)] = P_a + P_b = 1$, which implies that the trace of the auxiliary states perform a classical random walk.

From Eqs. (65) and (69) it becomes evident that the populations of the system remain unchanged during all the evolution. On the other hand, the dynamic of the coherences can be obtained straightforwardly in the Laplace domain. From Eq. (70) we get

$$\Phi_a^{\pm}(u) = h_{ab}(u)\Phi_S^{\pm}(0), \qquad \Phi_b^{\pm}(u) = h_{ba}(u)\Phi_S^{\pm}(0), \quad (72)$$

where we have introduced the auxiliary function

$$h_{ab}(u) = \frac{(P_a - P_b)\gamma_{ab} + P_a(u + \gamma_b)}{\gamma_{ba}(u + \gamma_a) + \gamma_{ab}(u + \gamma_b) + (u + \gamma_a)(u + \gamma_b)}. \quad (73)$$

Therefore, from Eq. (65) the matrix elements of $\rho_S(t)$ read

$$\Pi_S^{\pm}(t) = \Pi_S^{\pm}(0), \qquad \Phi_S^{\pm}(t) = h(t)\Phi_S^{\pm}(0), \quad (74)$$

where $h(t) = h_{ab}(t) + h_{ba}(t)$, gives the coherences decay. From these solutions, it is straightforward to obtain the corresponding system evolution

$$\frac{d\rho_S(t)}{dt} = \int_0^t d\tau K(t-\tau)\mathcal{L}[\rho_S(\tau)], \quad (75)$$

with $\mathcal{L}[\bullet] = (-\bullet + \sigma_z \bullet \sigma_z)$ and $K(u) = [1 - uh(u)]/h(u)$.

In order to check the completely positive condition, we write the solution map as

$$\rho_S(t) = g_+(t)\rho(0) + g_-(t)\sigma_z\rho(0)\sigma_z \quad (76)$$

FIG. 1: Normalized coherences $\Phi_S^{\pm}(t)/\Phi_S^{\pm}(0) = h(t)$, Eq. (74). In the upper curve the parameters are $\gamma_a = 0.1$, $\gamma_b = 1$, and $\gamma_{ab} = \gamma_{ba} = 0$. In the lower curve they are $\gamma_a = 0.1$, $\gamma_b = 1$, $\gamma_{ab} = 1$, and $\gamma_{ba} = 0.1$. The rates are expressed in arbitrary units (a.u.). In both curves we take $P_a = 0.1$ and $P_b = 0.9$.

with $g_{\pm}(t) = [1 \pm h(t)]/2$. This mapping is completely positive at all times if $g_{\pm}(t) \geq 0$ [2, 3, 4], and in turn implies the constraint

$$|h(t)| \leq 1. \quad (77)$$

In the upper curve of Fig. (1) we plot the normalized coherences $\Phi_S^{\pm}(t)/\Phi_S^{\pm}(0) = h(t)$ for the case in which the non-diagonal rates are null, $\gamma_{ab} = \gamma_{ba} = 0$. Then, the dynamics reduce to a superposition of exponential decays, each one participating with weights P_a and P_b.

In the lower curve of Fig. (1) the non-diagonal rates are non-null, while the rest of the parameters remain the same as in the upper curve. In contrast to the previous case, here the coherence decay develops an oscillatory behavior that attain negative values. Clearly, this regime is unreachable by a superposition of exponential decays.

In both cases, the condition Eq. (77) is satisfied, guaranteeing the physical validity of the respective solutions.

Stochastic representation

The evolution Eq. (66) admits a stochastic interpretation like that proposed previously. The stochastic trajectories can be simulated with the following algorithms. First, for being consistent with the initial condition, the *system initialization* must be realized as follows

i) Generate a random number $r \in (0, 1)$.

ii) If $r \leq P_a$ ($r > P_a$) the dynamic initialize in channel a (b) with $\tilde{\rho}_a(0) = \rho_S(0)$ [$\tilde{\rho}_b(0) = \rho_S(0)$].

Trivially, with this procedure the channel a (b) is initialized with probability P_a (P_b).

By comparing Eqs. (66) and (57), the scattering superoperator results $\mathcal{E}[\bullet] = \sigma_z \bullet \sigma_z$, which does not depends

on the channel (a and b). It action over an arbitrary state [Eq. (68)] is ($R = a, b$)

$$\mathcal{E}[\tilde{\rho}_R(t)] = \sigma_z \tilde{\rho}_R(t) \sigma_z = \begin{pmatrix} \Pi_R^+(t) & -\Phi_R^+(t) \\ -\Phi_R^-(t) & \Pi_R^-(t) \end{pmatrix}. \quad (78)$$

Therefore, its application implies a change of sign for the coherence components. On the other hand, the self-dynamics Eq. (52) of each channel is defined by $\bar{\mathcal{L}}_{a/b}[\bullet] = \gamma_{a/b}(-\bullet + \sigma_z \bullet \sigma_z)$.

With the previous information, the *single trajectories* can be constructed with the following algorithm:

1) Given that the system has arrived at time t_i to channel a, generate a random number $r \in (0, 1)$ and solve for $(t_{i+1} - ti)$ from the equation $P_0^{(a)}(t_{i+1} - ti) = r$, where $P_0^{(a)}(t) = \exp[-\gamma_{ba} t]$.

2) For times satisfying $t \in (t_{i+1}, ti)$, the dynamics in channel a is defined by its self-propagator, $\tilde{\rho}_a(t) = \exp[(t - t_i)\bar{\mathcal{L}}_a]\tilde{\rho}_a(t_i)$.

3) At time t_{i+1} the system is transferred from channel a to b, implying the transformation $\tilde{\rho}_b(t_{i+1}) \to \mathcal{E}[\tilde{\rho}_a(t_{i+1})]$ and the posterior resetting of channel a, defined by $\tilde{\rho}_a(t_{i+1}) \to 0$.

4) Go to 1) with $a \leftrightarrow b$ and $i \to i + 1$.

At this point, it is immediate to realize that the classical rate equations Eqs. (69) and (71) arise straightforwardly from the (transfer) jumps between both channels. The corresponding stationary traces read

$$\mathrm{Tr}[\tilde{\rho}_a(\infty)] = \frac{\gamma_{ab}}{\gamma_{ab} + \gamma_{ba}}, \quad \mathrm{Tr}[\tilde{\rho}_b(\infty)] = \frac{\gamma_{ba}}{\gamma_{ab} + \gamma_{ba}}, \quad (79)$$

which do not depend on the system initial state.

In contrast with the population evolution, some non-standard dynamical properties can be found in the coherences evolution when $\gamma_a = \gamma_b = 0$. In Fig. (2) we show the normalized coherences $\Phi_S^{\pm}(t)/\Phi_S^{\pm}(0) = h(t)$ corresponding to this case. In the inset, it is shown a typical stochastic realization of the coherences of the auxiliary matrixes $\tilde{\rho}_a(t)$ and $\tilde{\rho}_b(t)$ obtained with the previous algorithm. As expected, in each application of $\mathcal{E}$ the coherences are transferred between both channels with a change of sign. We also show an average over 500 realizations. We checked that by increasing the number of realizations, the average behavior result indistinguishable with the dynamics Eq. (74).

In strong contrast with the previous figure, in Fig. (2) the stationary values of the coherences are "not null and depend on the initial condition." In fact, their normalized asymptotic value is $\lim_{t\to\infty} \Phi_S^{\pm}(t)/\Phi_S^{\pm}(0) \simeq -0.654$. This characteristic is consistent with the breakdown of condition Eq. (63) and can be understood in terms of our previous analysis. By taking $\gamma_a = \gamma_b = 0$ in Eq. (72) we get

$$\Phi_a^{\pm}(u) = \frac{P_a(u + \gamma_{ab}) - P_b \gamma_{ab}}{u[u + \gamma_{ab} + \gamma_{ba}]} \Phi_S^{\pm}(0), \quad (80)$$

FIG. 2: Normalized coherences $\Phi_S^{\pm}(t)/\Phi_S^{\pm}(0) = h(t)$, Eq. (74). The parameters are $\gamma_a = \gamma_b = 0$, $\gamma_{ab} = 1$, $\gamma_{ba} = 0.1$, with the statistical weights $P_a = 0.1$ and $P_b = 0.9$. The noisy curve correspond to an average over 500 realizations of the trajectories defined in the text. The inset show a particular realization for the coherences $\Phi_a^{\pm}(t)$ and $\Phi_b^{\pm}(t)$ of the auxiliary matrixes $\tilde{\rho}_a(t)$ and $\tilde{\rho}_b(t)$ respectively.

which implies the asymptotic value

$$\lim_{t\to\infty} \Phi_a^{\pm}(t) = (P_a - P_b)\frac{\gamma_{ab}}{\gamma_{ab} + \gamma_{ba}} \Phi_S^{\pm}(0), \quad (81a)$$

$$= (P_a - P_b)\mathrm{Tr}[\tilde{\rho}_a(\infty)]\Phi_S^{\pm}(0). \quad (81b)$$

This last expression can be easily interpreted in terms of the realizations of the proposed stochastic dynamics. From the inset of Fig. (2), it is clear that, in spite of a change of sign, the coherence transferred between both channels does not change along all the evolution. In fact, notice that due to the election $\gamma_a = \gamma_b = 0$, the self-propagators of both channels [see previous step 2)] are the identity operator. Therefore, all realizations that begin in channel a [measured by P_a] that are found in channel a in the stationary regime (measured by $\mathrm{Tr}[\tilde{\rho}_a(\infty)]$), contributes to the stationary value of the coherence $\Phi_a^{\pm}(t)$ with the value $\Phi_S^{\pm}(0)$. This argument explain the contribution proportional to $P_a \mathrm{Tr}[\tilde{\rho}_a(\infty)]\Phi_S^{\pm}(0)$ in Eq. (81). On the other hand, a similar contribution is expected from the realizations that begin in channel b. Nevertheless, due to the action of the superoperator $\mathcal{E}$ [Eq. (78)] they contributes with the opposite sign.

By adding the contributions of both auxiliary matrixes, from Eq. (81) the stationary system coherences reads

$$\lim_{t\to\infty} \Phi_S^{\pm}(t) = (P_a - P_b)\left\{\frac{\gamma_{ab} - \gamma_{ba}}{\gamma_{ab} + \gamma_{ba}}\right\} \Phi_S^{\pm}(0) \neq 0, \quad (82)$$

This expression fits the stationary value of Fig. (2).

The stochastic realizations corresponding to the system coherence $\Phi_S^{\pm}(t)$ can be trivially obtained from the the realizations of $\Phi_a^{\pm}(t)$ and $\Phi_b^{\pm}(t)$. By adding the upper and lower realizations of the inset of Fig. (2), we get a function that fluctuates between the values $\pm\Phi_S^{\pm}(0)$.

By considering the initial conditions and the superoperator action from these realizations it is also possible to understand the four contribution terms of Eq. (82). Finally, we remark that when any of both channels have a non-trivial self-dynamics, the coherences vanish in the stationary regime, losing any dependence on the system initial condition $\rho_S(0)$ [see Fig. (1)].

B. Depolarizing reservoir

Another example that admits a stochastic representation is the case of a depolarizing reservoir [4, 15], which is defined by the superoperator

$$\mathcal{E}[\bullet] = (\sigma_x \bullet \sigma_x + \sigma_y \bullet \sigma_y)/2, \tag{83}$$

where σ_x and σ_y are the x and y Pauli matrixes respectively. For simplifying the analysis we assume channels without self-dynamics. Therefore, the evolution reads

$$\frac{d}{dt}\tilde{\rho}_a(t) = -\gamma_{ba}\tilde{\rho}_a(t) + \gamma_{ab}\mathcal{E}[\tilde{\rho}_b(t)], \tag{84a}$$

$$\frac{d}{dt}\tilde{\rho}_b(t) = -\gamma_{ab}\tilde{\rho}_b(t) + \gamma_{ba}\mathcal{E}[\tilde{\rho}_a(t)]. \tag{84b}$$

The action of the superoperator $\mathcal{E}$ over the states $\tilde{\rho}_R(t)$ [Eq. (68)] is given by ($R = a, b$)

$$\mathcal{E}[\tilde{\rho}_R(t)] = \begin{pmatrix} \Pi_R^-(t) & 0 \\ 0 & \Pi_R^+(t) \end{pmatrix}. \tag{85}$$

Therefore, its application destroy the coherences components and interchange the populations of the upper and lower states.

The populations of the auxiliary states evolve as

$$\frac{d}{dt}\Pi_a^+(t) = -\gamma_{ba}\Pi_a^+(t) + \gamma_{ab}\Pi_b^-(t), \tag{86a}$$

$$\frac{d}{dt}\Pi_b^-(t) = -\gamma_{ab}\Pi_b^-(t) + \gamma_{ba}\Pi_a^+(t), \tag{86b}$$

subject to the initials conditions $\Pi_a^+(0) = P_a\Pi_S^+(0)$ and $\Pi_b^-(0) = P_b\Pi_S^-(0)$. The evolution of $\Pi_b^+(t)$ and $\Pi_a^-(t)$ follows after changing $a \leftrightarrow b$. Notice that this splitting of the population couplings follows from the superoperator action defined by Eq. (85). On the other hand, the coherences evolution read

$$\frac{d}{dt}\Phi_a^\pm(t) = -\gamma_{ba}\Phi_a^\pm(t), \qquad \frac{d}{dt}\Phi_b^\pm(t) = -\gamma_{ab}\Phi_b^\pm(t). \tag{87}$$

Therefore, in this case the stationary coherences are null. This fact also follows trivially from Eq. (85). In contrast, the stationary populations reads

$$\Pi_a^+(\infty) = [\Pi_S^+(0)P_a + \Pi_S^-(0)P_b]\frac{\gamma_{ab}}{\gamma_{ab}+\gamma_{ba}}, \tag{88a}$$

$$\Pi_b^-(\infty) = [\Pi_S^+(0)P_a + \Pi_S^-(0)P_b]\frac{\gamma_{ba}}{\gamma_{ab}+\gamma_{ba}}, \tag{88b}$$

where $\Pi_b^+(\infty)$ and $\Pi_a^-(\infty)$ follows after changing $a \leftrightarrow b$. This result has an immediate interpretation in the context of the stochastic approach. In fact, the last fractional factors correspond to the "natural" stationary solutions of Eq. (86). This solution is corrected by the terms in brackets, which in fact take in account the system initialization [notice that $\Pi_a^+(0) + \Pi_b^-(0) \neq 1$] and the transformations induced by the superoperator $\mathcal{E}$ Eq. (85). Finally, the system stationary populations $\Pi_S^\pm(\infty) = \Pi_a^\pm(\infty) + \Pi_b^\pm(\infty)$ reads

$$\Pi_S^\pm(\infty) = \Pi_S^\pm(0)\frac{P_a\gamma_{ab} + P_b\gamma_{ba}}{\gamma_{ab}+\gamma_{ba}} + \Pi_S^\mp(0)\frac{P_a\gamma_{ba} + P_b\gamma_{ab}}{\gamma_{ab}+\gamma_{ba}}. \tag{89}$$

As in the previous case, the dependence of the stationary state in the initial conditions is lost when the channels have a proper dissipative self-dynamics.

IV. SUMMARY AND CONCLUSIONS

We have presented a new class of dynamical master equations that provide an alternative framework for the characterization of non-Markovian open quantum system dynamics. In this approach, the system state is written in terms of a set of auxiliary matrixes whose evolutions involve Lindblad contributions with coupling between all of them, resembling the structure of a classical rate equation.

We have derived the previous structure from different approaches. In the context of the GBMA, a complex structured reservoir is approximated in terms of a direct sum of Markovian sub-reservoirs. Then, the Lindblad rate structure arises by considering arbitrary interaction Hamiltonians that couple the different subspaces associated to each sub-reservoir. The matrix structures that define the system evolution are expressed in terms of the projected bath correlations.

On the other hand, we have derived the same structure from composite environments, where the entanglement between the system and a Markovian environment is modulated by extra unobserved degrees of freedom. The Lindblad rate structure arises straightforwardly when the tripartite interaction Hamiltonian that involve the three parts does not couple the coherences and populations of the extra degrees of freedom. This scheme also allows to find the conditions under which an arbitrary Lindblad rate equation provides a completely positive evolution.

Due to the apparent similarity of the evolution with a classical rate equation, we have also formulated a quantum stochastic dynamics that in average is described by a Lindblad rate equation. The stochastic dynamic consists in a set of transmission channels, each one endowed with a different self-system evolution, and where the transitions between them are attended by the application of a completely positive superoperator. This formalism allows to understand some amazing properties of the non-Markovian dynamics, such as the dependence of the sta-

tionary state in the initial conditions. This phenomenon arise from the interplay between the initial channel occupations and the structure of the stochastic dynamics. We exemplified our results by analyzing the dynamical action of non-trivial complex dephasing and depolarizing reservoirs over a single qubit system.

In conclusion, we have presented a close formalism that defines an extra class of non-Markovian quantum processes that may be of help for understanding different physical situations where the presence of non-local effects is relevant [24, 25, 26, 27, 28, 29, 30, 31, 32].

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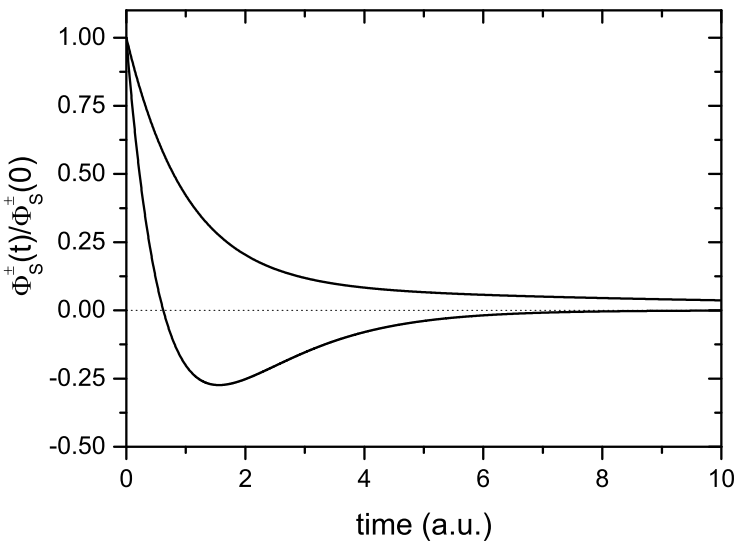
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0.75
0.50
0.25
0.00
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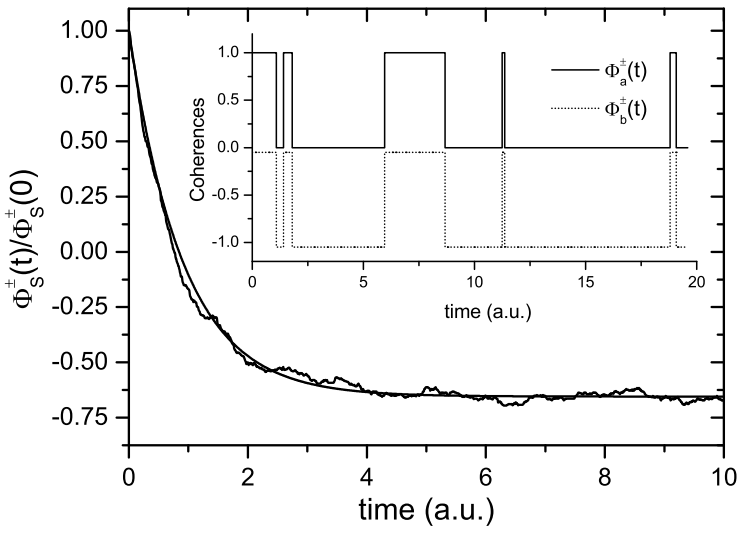
-0.50
0
2
4
6
8
10
time (a.u.)
$\Phi_S^{\pm}(t)/\Phi_S^{\pm}(0)$


1.00
0.75
0.50
0.25
0.00
-0.25
-0.50
-0.75
0
2
4
6
8
10
time (a.u.)
$\Phi_S^{\pm}(t)/\Phi_S^{\pm}(0)$
Coherences
1.0
0.5
0.0
-0.5
-1.0
0
5
10
15
20
time (a.u.)
$\Phi_a^{\pm}(t)$
$\Phi_b^{\pm}(t)$